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ON CALABI EXTREMAL KÄHLER-RICCI SOLITONS

SIMONE CALAMAI AND DAVID PETRECCA

ABSTRACT. In this note we give a characterization of Kähler metrics which are both Calabi extremal and Kähler-Ricci solitons in terms of complex Hessians and the Riemann curvature tensor. We apply it to prove that, under the assumption of positivity of the holomorphic sectional curvature, these metrics are Einstein.

INTRODUCTION

Let M^{2n} be a compact complex manifold. A Kähler metric g on M is said to be *Kähler-Einstein* if it is Einstein as a Riemannian metric, i.e. it is proportional to its Ricci tensor or, equivalently, if there exists $c \in \mathbb{R}$ such that

$$(1) \quad \rho_g = c\omega_g$$

where ρ_g (resp. ω_g) denotes the Ricci form (resp. Kähler form) of g .

There are two possible generalizations of this notion. The first is the notion of *extremal metric* introduced by Calabi in [5, 6] (see also [14]) in the attempt to find a canonical representative in a given Kähler class $\Omega \in H^{1,1}(M) \cap H^2(M, \mathbb{R})$. These metrics are defined to be the critical points of the Riemannian functional $\mathcal{M}_\Omega \rightarrow \mathbb{R}$ defined by

$$g \longmapsto \int_M s_g^2 \omega_g^n$$

where \mathcal{M}_Ω is the space of the Kähler metrics on M in the class Ω and s_g is the scalar curvature of g . He also showed that a metric is extremal if, and only if, the gradient of s_g is a holomorphic vector field. Constant scalar curvature Kähler metrics (cscK), hence in particular Kähler-Einstein, are examples of extremal metrics, but there are extremal metrics of non constant scalar curvature (see again [5]).

Another direction to generalize the Einstein condition (1) is the following. A Kähler metric g is called a *Kähler-Ricci soliton* (KRS) if there exist $c \in \mathbb{R}$ and a holomorphic vector field X such that

$$\rho_g + c\omega_g = \mathcal{L}_X \omega_g.$$

These metrics give rise to special solutions of the Kähler-Ricci flow (see e.g. [9]), namely they evolve under biholomorphisms. It is known that on a compact manifold, if $c \geq 0$ then g is Einstein (see e.g. again [9]), so in the compact Kähler case one only considers the so-called *shrinking* Kähler-Ricci solitons ($c < 0$) whose equation, after a scaling, can be written as

$$(2) \quad \rho_g - \omega_g = \mathcal{L}_X \omega_g.$$

The Hodge decomposition for the dual of X allows us to introduce a holomorphy potential with respect to g , i.e. a complex-valued function θ_X such that $\iota_X \omega_g =$

$i\bar{\partial}\partial\theta_X$. By means of this function we can infer that the Kähler form ω_g belongs to $2\pi c_1(M)$, making M a Fano manifold.

The first examples of non-Einstein compact Kähler-Ricci solitons go back to the constructions of Koiso [17] and independently Cao [7] of Kähler metrics on certain \mathbb{CP}^1 -bundles over \mathbb{CP}^n . Koiso himself remarks that this Kähler-Ricci soliton metric is not Calabi extremal and proves that if it were, it would be Einstein.

There is a class of manifolds for which there are existence results for both kinds of metrics, namely toric manifolds (see e.g. [1]). For extremal metrics we mention for instance the existence result of Donaldson [10] for toric surfaces. For the KRS we refer to the existence result, in all dimensions, of Wang and Zhu [21]. Finally, the existence of a Kähler-Einstein metric on a compact Fano manifold is equivalent to the notion of *K-stability* stated by Chen, Donaldson and Sun in [8] and subsequent papers.

It is then natural to ask what happens when a metric generalizes a Kähler-Einstein metric in both these ways. In Theorem 2.7 we prove, under the assumption of positivity of the holomorphic sectional curvature, that an extremal KRS is in fact Kähler-Einstein.

It is not too restrictive to assume positive holomorphic sectional curvature provided it does not exceed a certain numerical bound. Indeed it has been proved by Futaki and Sano [13] that if the diameter of a Ricci soliton is $< \frac{10}{13}\pi$ then the soliton is trivial. On the other hand, Tsukamoto [20] proved that if a Kähler metric g has holomorphic sectional curvature $> \varepsilon$, then the diameter of the manifold is bounded from above by $\frac{\pi}{\sqrt{\varepsilon}}$.

From these results we can infer that if the holomorphic sectional curvature is greater than $(\frac{13}{10})^2$ then the KRS must be Einstein, so our result is non trivial when the holomorphic sectional curvature does not exceed this number. The authors do not know whether there are any connections between positive holomorphic sectional curvature and the extremality condition or whether the extremality gives conditions on the diameter .

The paper is organized as follows. We start recalling some notation and conventions of Kähler geometry. This level of detail seems necessary in order to avoid confusion among different conventions. We then go on proving, for a Kähler-Ricci soliton, the characterization of being extremal in terms of the length of the complex Hessian of its potential function and in terms of certain contractions of the Riemann curvature tensor. We then use this to prove our main result in Theorem 2.7 and we also give a condition about the isometry group of a non-Einstein extremal KRS. We finally make a remark about the replicability of the argument in the Sasakian setting.

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1. DEFINITIONS AND PRELIMINARY RESULTS

Notation. Let (M, g, J) be a smooth, compact, without boundary Kähler manifold of real dimension $2n$, with its Riemannian metric g and compatible integrable complex structure J . The corresponding Kähler form is $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$. We also

denote as Ric the Riemannian Ricci tensor corresponding to the Riemannian metric g ; and its Ricci form as $\rho(\cdot, \cdot) = \text{Ric}(J\cdot, \cdot)$. We label s the Riemannian scalar curvature of the metric g . We let δ be adjoint of the exterior differential d with respect to g and $\Delta_d = \delta d + d\delta$ be the d -Laplacian acting on differential forms.

We let \sharp and \flat denote the musical isomorphisms between fields and 1-forms. For a 1-form α we denote as $|\alpha|^2 = (\alpha, \alpha) = (\alpha^\sharp, \alpha^\sharp)$, and as well as $|Z|^2 = (Z, Z) = (Z^\flat, Z^\flat)$ the metric pairing on by means of the Riemannian metric g . Similarly, if a real $(1, 1)$ -form β and a 2-tensor B correspond each other via $\beta(\cdot, \cdot) = B(J\cdot, \cdot)$, then we have for the metric pairings $|B|^2 = (B, B) = 2|\beta|^2 = 2(\beta, \beta)$. For example, $|\text{Ric}|^2 = (\text{Ric}, \text{Ric}) = 2|\rho|^2 = 2(\rho, \rho)$. Notice also that for any smooth real valued function on M there holds $(\omega, dd^c u) = -\Delta_d u$.

Given any tensor T and any vector field V on a smooth manifold, we label as $\mathcal{L}_V T$ the Lie derivative of T along V .

For any smooth, real valued function u on M , we label as ∇u the Riemannian gradient of u ; namely, $\nabla u = (du)^\sharp$. We also denote its $(1, 0)$ -part as $\partial^\sharp f = \frac{1}{2}(\nabla f - iJ\nabla f)$. We let $\text{Hess } u = \frac{1}{2}\mathcal{L}_{\nabla u} g$ be the real Hessian of u .

We also label as $\mathfrak{h}(M)$ the algebra of (complex) holomorphic vector fields of M .

The first definition is very classical and tracks back Hamilton [16].

Definition 1.1. Let (M, g, J) and Ric as in *Notation*; let f be a smooth, real valued function on M . We say that (g, f) is a Kähler-Ricci soliton when the following equation is satisfied

$$(3) \quad \text{Ric} - g = \frac{1}{2}\mathcal{L}_{\nabla f} g.$$

Remark 1.2. It is a general fact that ∇f is real holomorphic, although this is often stated in the definition. Indeed, equation (3) implies that $\nabla_i \nabla_j f = 0$ for all $i, j \in \{1, \dots, n\}$.

The next definition is due to Calabi [5].

Definition 1.3. Let (M, g, J) and s be as in *Notation*. We say that the metric g is *Calabi extremal*, or simply extremal, when the Riemannian gradient of s is holomorphic, i.e. if $\bar{\partial}\partial^\sharp s = 0$.

In this paper we consider metrics which satisfy both these definitions.

Definition 1.4. Let (M, g, J) as in *Notation*, and let (g, f) be a Kähler-Ricci soliton as in Definition 1.1. Moreover, let g be Calabi extremal as in Definition 1.3. Then, we call (g, f) an extremal Kähler-Ricci soliton.

Remark 1.5. We chose to label the pairs (g, f) in Definition 1.4 as extremal Kähler-Ricci solitons although a similar name was given by Guan in [15] to different objects.

Not all complex valued smooth functions v on M give rise to holomorphic vector fields. The ones which do are solutions of the equation $\bar{\partial}\partial^\sharp v = 0$, they lie in the kernel of the fourth order differential operator $L_g = (\bar{\partial}\partial^\sharp)^* \bar{\partial}\partial^\sharp$ (see [11]).

The presence of an extremal metric gives information about the algebra of holomorphic vector fields $\mathfrak{h}(M)$. Namely the following theorems hold.

Theorem 1.6 ([6, 11]). *Let g be an extremal Kähler metric on M with scalar curvature s . Then the Lie algebra $\mathfrak{h}(M)$ has a semidirect sum decomposition*

$$(4) \quad \mathfrak{h}(M) = \mathfrak{a}(M) \oplus \mathfrak{h}'(M),$$

where $\mathfrak{a}(M)$ is the complex Lie subalgebra of $\mathfrak{h}(M)$ consisting of all parallel holomorphic vector fields of M , and $\mathfrak{h}'(M)$ is an ideal of $\mathfrak{h}(M)$ consisting of the image under ∂^\sharp of the kernel of L_g .

Moreover $\mathfrak{h}'(M)$ has a decomposition

$$\mathfrak{h}'(M) = \bigoplus_{\lambda \geq 0} \mathfrak{h}_\lambda(M)$$

where $[\partial^\sharp s, Y] = \lambda Y$ for any $Y \in \mathfrak{h}(M)$. Furthermore the centralizer $\mathfrak{h}_0(M)$ of $\partial^\sharp s$ is the complexification of the Lie algebra consisting of Killing vector fields of M .

In the case of a Kähler-Ricci soliton a similar theorem holds.

Theorem 1.7 ([19]). *If g is a Kähler-Ricci soliton with $(1, 0)$ -vector field X . Then $\mathfrak{h}(M)$ admits the decomposition*

$$(5) \quad \mathfrak{h}(M) = \mathfrak{k}_0(M) \oplus \bigoplus_{\lambda > 0} \mathfrak{k}_\lambda(M),$$

where $\mathfrak{k}_\lambda(M) = \{Y \in \mathfrak{h}(M) : [X, Y] = \lambda Y\}$. Moreover the centralizer $\mathfrak{k}_0(M)$ of X splits as $\mathfrak{k}'_0 \oplus \mathfrak{k}''_0$ where \mathfrak{k}'_0 is the ∂^\sharp -image of real functions and \mathfrak{k}''_0 is the ∂^\sharp -image of purely imaginary functions.

The following result is due to Lichnerowicz (see [3, Proposition 2.140]).

Proposition 1.8. *On a compact Kähler manifold a (real) vector field X is holomorphic if and only if*

$$(6) \quad \Delta_d X^\flat - 2 \operatorname{Ric}(X, \cdot) = 0.$$

2. STATEMENTS AND PROOFS

The function f in Definition 1.1 has, by means of Remark 1.2, holomorphic gradient so it satisfies, applying δ on both sides of (6) (cf. [14, (1.17.5)])

$$(7) \quad \frac{1}{2} \Delta_d^2 f + (dd^c f, \omega + \frac{1}{2} dd^c f) + \frac{1}{2} (ds, df) = 0.$$

Tracing the KRS equation (3) we get $2n - s = \Delta_d f$, we substitute it into (7) to get

$$\frac{1}{2} \Delta_d (2n - s) - \Delta_d f + \frac{1}{2} |dd^c f|^2 + \frac{1}{2} (ds, df) = 0.$$

So

$$(8) \quad -\Delta_d s + 2(s - 2n) + |dd^c f|^2 + (ds, df) = 0.$$

Differentiating we get

$$(9) \quad -\Delta_d ds + 2ds + d|dd^c f|^2 + d(ds, df) = 0.$$

The last term in (9) can be substituted with the following two lemmas.

Lemma 2.1. *On an extremal Kähler-Ricci soliton (g, f) the holomorphic fields ∇f and ∇s commute.*

Proof. Since (g, f) is an extremal Kähler-Ricci soliton, then both $\partial^\sharp s$ and $\partial^\sharp f$ are holomorphic vector fields, i.e. $\partial^\sharp s, \partial^\sharp f \in \mathfrak{h}(M)$. Also, by means of Theorem 1.6, $\mathfrak{h}(M)$ splits as $\mathfrak{h}(M) = \mathfrak{a}(M) \oplus \bigoplus_{\lambda \geq 0} \mathfrak{h}_\lambda(M)$. The summand $\mathfrak{h}_0(M)$, the centralizer of $\partial^\sharp s$ in $\mathfrak{h}(M)$, contains $\partial^\sharp f$. Hence we have

$$\begin{aligned} 0 &= [\partial^\sharp s, \partial^\sharp f] = \frac{1}{4} ([\nabla s, \nabla f] - [J\nabla s, J\nabla f] - i[\nabla s, J\nabla f] - i[J\nabla s, \nabla f]) \\ &= \frac{1}{2} ([\nabla s, \nabla f] - iJ[\nabla s, \nabla f]) , \end{aligned}$$

and we take its real part to conclude. \square

Lemma 2.2. *Whenever two real functions u, v satisfy $[\nabla u, \nabla v] = 0$, then there holds*

$$d(g(\nabla u, \nabla v)) = (\nabla_{\nabla u} \nabla v + \nabla_{\nabla v} \nabla u)^\flat = 2(\nabla_{\nabla u} \nabla v)^\flat .$$

Proof. For any vector field Y we have

$$\begin{aligned} Y \cdot g(\nabla v, \nabla u) &= g(\nabla_Y \nabla v, \nabla u) + g(\nabla v, \nabla_Y \nabla u) \\ &= g(\nabla_{\nabla u} \nabla v + \nabla_{\nabla v} \nabla u, Y) \\ &= g(2\nabla_{\nabla u} \nabla v, Y) . \end{aligned}$$

This completes the proof of the lemma. \square

For a (Kähler-)Ricci soliton there are some quantities that are constant, see e.g. [9]. One of them is, in our notation,

$$(10) \quad s + |\nabla f|^2 + 2f = \text{const} .$$

From this together with Lemma 2.2 it is easy to infer the following.

Lemma 2.3. *Let g be a KRS with real holomorphic field X and let $Z = X^{1,0}$. Then g is extremal if, and only if, $\nabla_X X$ is real holomorphic (or $\nabla_Z Z$ is holomorphic or $\nabla_{\bar{Z}} \bar{Z}$ is antiholomorphic).*

At this point it is worth noticing the following.

Proposition 2.4. *For an extremal KRS g with field X and scalar curvature s , if $X = c\nabla s$ then g is Einstein.*

Proof. We first notice that c has to be constant on M . Indeed if it were a function on M it would be holomorphic since the two fields are. Consider the function $p \mapsto |X_p|^2$ and a local maximum $q \in M$. At q we would have

$$g_q(\nabla_X X|_q, X_q) = 0 .$$

Under the proportionality assumption (10) becomes

$$(11) \quad (c + 2)X + 2\nabla_X X = 0 .$$

At q we would have then $\frac{c+2}{2}g_q(X, X) = 0$ which implies $X = 0$ if $c \neq -2$.

If $c = -2$ we have from (11) and Lemma 2.2 that $\nabla_X X = \nabla|X|^2 = 0$ implying $X = 0$ as well. \square

By means of the decomposition theorems 1.6 and 1.7, the fields JX and $J\nabla s$ belong to the center of the isometry algebra and are linearly independent for a non-Einstein extremal KRS. This gives us the following corollary.

Corollary 2.5. *If g is a non-Einstein extremal KRS, then the center of the isometry group of g has dimension at least 2.*

We now present a characterization of extremal Kähler-Ricci solitons.

Proposition 2.6. *Let (M^{2n}, g, ω, f) be a compact Kähler-Ricci soliton with Riemannian scalar curvature s . Let $X = \nabla f$. Then the following are equivalent.*

- (1) *the function $|dd^c f|^2$ is constant and $[\nabla f, \nabla s] = 0$;*
- (2) *g is extremal;*
- (3) *$\text{Rm}(\cdot, \overline{\partial^\sharp f}) \overline{\partial^\sharp f} = 0$;*
- (4) *The tensor*

$$T_X := \text{Rm}(\cdot, X)X$$

commutes with J and $\alpha : (A, B) \mapsto \text{Rm}(A, JX, X, B)$ is a $(1, 1)$ -form.

Proof. Let us first prove the equivalence between (1) and (2). By means of Proposition 1.8, the condition on g being extremal is equivalent to require the Riemannian scalar curvature s to satisfy the tensorial Lichnerowicz equation (see [3, Proposition 2.140])

$$(12) \quad \Delta_d s - 2 \text{Ric}(\nabla s, \cdot) = 0.$$

Let us assume g to be extremal. Equation (12), together with the Kähler-Ricci soliton assumption $\text{Ric} = g + \text{Hess } f$, reads

$$(13) \quad \begin{aligned} 0 &= \Delta_d s - 2ds - 2 \text{Hess}_f(\nabla s, \cdot) \\ &= \Delta_d s - 2ds - 2g(\nabla_{\nabla s} \nabla f, \cdot). \end{aligned}$$

By means of Lemma 2.2, formula (13) differs from (9) by the term $d|dd^c f|^2$ which has to be zero.

Conversely, assuming $[\nabla s, \nabla f] = 0$, then Lemma 2.2 holds. Also, in (9) the term $d|dd^c f|^2$ vanishes and then (9) is simply (12), which says, by means of Proposition 1.8, that s has holomorphic gradient. This completes the equivalence between (1) and (2).

Let g be extremal, then by means of the previous Lemma, the field $\nabla_Z Z$ where $Z = \partial^\sharp f$ is holomorphic. Then compute for any $(1, 0)$ -field A ,

$$\begin{aligned} \text{Rm}(A, \overline{Z})\overline{Z} &= \nabla_A \nabla_{\overline{Z}} \overline{Z} - \nabla_{\overline{Z}} \nabla_A \overline{Z} - \nabla_{[\overline{Z}, A]} \overline{Z} \\ &= 0 \end{aligned}$$

by using the fact that \overline{Z} and $\nabla_{\overline{Z}} \overline{Z}$ are antiholomorphic (hence killed by ∇_A) that kills the first two terms and that $[\overline{Z}, A]$ is $(1, 0)$ that kills the last.

Conversely, the generic form of the Riemann tensor for A of type $(1, 0)$ is

$$\text{Rm}(A, \overline{Z})\overline{Z} = \nabla_A \nabla_{\overline{Z}} \overline{Z}.$$

If this is zero, it means that the field $\nabla_{\overline{Z}} \overline{Z}$ is killed by ∇_A for any A of type $(1, 0)$ implying that it is antiholomorphic. Indeed, for any W we have

$$0 = \nabla_{Y-iJY}(W + iJW) = \nabla_Y W + J\nabla_{JY} W + i(J\nabla_Y W - \nabla_{JY} W)$$

that is, W satisfies $\nabla_{JY} W = J\nabla_Y W$ for all Y , that is W is real holomorphic.

Hence we conclude g is extremal by means of the previous Lemma.

Let us now assume (3). We notice that its real formulation is given by the system

$$(14) \quad \begin{cases} \text{Rm}(A, X)JX = -\text{Rm}(A, JX)X \\ \text{Rm}(A, X)X = \text{Rm}(JA, X)JX. \end{cases}$$

and the second equation means exactly that $[T_X, J] = 0$. We have now, using (14) for the second equality,

$$\begin{aligned} \alpha(B, A) &= \text{Rm}(B, JX, X, A) \\ &= -\text{Rm}(B, X, JX, A) \\ &= -\text{Rm}(JX, A, B, X) \\ &= -\alpha(A, B). \end{aligned}$$

To prove that α is $(1, 1)$ is equivalent to prove that it is J -invariant. This follows again from (14) since

$$\begin{aligned} \alpha(JA, JB) &= \text{Rm}(JA, JX, X, JB) \\ &= -\text{Rm}(A, X, JX, B) \\ &= \text{Rm}(A, JX, X, B) \\ &= \alpha(A, B). \end{aligned}$$

Conversely let $[T_X, J] = 0$ and let α be J -invariant. These assumption are exactly (14). \square

We can use this to prove our main result.

Theorem 2.7. *Any extremal Kähler-Ricci soliton with positive holomorphic sectional curvature is Einstein.*

Proof. Let f be the soliton function. Assume it is not a constant and Z be the normalized $\partial^\sharp f$.

By assumption we have, in the direction Z , that the holomorphic sectional curvature is

$$K(Z) := \text{Rm}(Z, \overline{Z}, Z, \overline{Z}) > 0.$$

By means of the previous proposition we are lead to the contradiction $K(Z) = 0$ as the above Riemann tensor vanishes. \square

Remark 2.8. There is no loss of generality to assume the positivity of the holomorphic sectional curvature instead of just requiring it to have a sign. Indeed, by a theorem of Berger [2, Lemme (7.4) pag. 50] prescribing the sign of the holomorphic sectional curvature gives the same sign to the scalar curvature, which in case of Ricci solitons is always positive by means of general results (see e.g. again [9]).

The argument exposed in this paper can be replicated verbatim to prove the following result about Sasaki manifolds. We refer to [12, 4] for the notions of Sasaki-Ricci solitons, to and Sasaki-extremal metrics and for transverse curvature. Recall that, for a Sasaki manifold, being transversally Kähler-Einstein is equivalent to being η -Sasaki-Einstein.

Theorem 2.9. *Any extremal Sasaki-Ricci soliton with positive transverse holomorphic sectional curvature is η -Sasaki-Einstein.*

Indeed there are Sasakian analogues of Theorem 4 done by Boyer and Galicki and an extension of Theorem 1.7 done by the second author in [18]. Moreover, the Lichnerowicz equations hold as well for the transverse quantities, see again [4].

REFERENCES

- [1] M. Abreu, *Kähler geometry of toric manifolds in symplectic coordinates*, Symplectic and contact topology: interactions and perspectives (Toronto, ON/Montreal, QC, 2001), Fields Inst. Commun., vol. 35, Amer. Math. Soc., Providence, RI, 2003, pp. 1–24.
- [2] M. Berger, *Sur les variétés d'Einstein compactes*, Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d'Expression Latine (Namur, 1965), Librairie Universitaire, Louvain, 1966, pp. 35–55.
- [3] A. Besse, *Einstein manifolds*, Springer Verlag, 1987.
- [4] C. Boyer, K. Galicki, and S. R. Simanca, *Canonical Sasakian metrics*, Commun. Math. Phys. **279** (2008), 705–733.
- [5] E. Calabi, *Extremal Kähler metrics*, Seminar on Differential Geometry, Ann. of Math. Stud., vol. 102, Princeton Univ. Press, Princeton, N.J., 1982, pp. 259–290.
- [6] ———, *Extremal Kähler metrics. II*, Differential geometry and complex analysis, Springer, Berlin, 1985, pp. 95–114.
- [7] H.-D. Cao, *Existence of gradient Kähler-Ricci solitons*, Elliptic and parabolic methods in geometry (Minneapolis, MN, 1994), A K Peters, Wellesley, MA, 1996, pp. 1–16.
- [8] S. Chen, S. K. Donaldson, and S. Sun, *Kähler-Einstein metrics and stability*, Int. Math. Res. Not., 2012.
- [9] B. Chow et al., *The Ricci flow: Techniques and applications: Geometric aspects*, Mathematical surveys and monographs, vol. 135, American Mathematical Society, 2007.
- [10] S. K. Donaldson, *Extremal metrics on toric surfaces: a continuity method*, J. Diff. Geom. **79** (2008), no. 3, 389–432.
- [11] A. Futaki, *Kähler-Einstein metrics and integral invariants*, Lecture Notes in Mathematics, no. 1314, Springer-Verlag, 1988.
- [12] A. Futaki, H. Ono, and G. Wang, *Transverse Kähler geometry of Sasaki manifolds and toric Sasaki-Einstein manifolds*, J. Diff. Geom. **83** (2009), 585–635.
- [13] A. Futaki and Y. Sano, *Lower diameter bounds for compact shrinking ricci solitons*, Asian J. Math. **17** (2013), no. 1, 17–32.
- [14] P. Gauduchon, *Calabi's extremal Kähler metrics: An elementary introduction*, To appear.
- [15] D. Guan, *Extremal solitons and exponential C^∞ convergence of the modified Calabi flow on certain \mathbb{CP}^1 bundles*, Pacific J. Math. **233** (2007), no. 1, 91–124.
- [16] R. S. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz, CA, 1986), Contemp. Math., vol. 71, Amer. Math. Soc., Providence, RI, 1988, pp. 237–262. MR 954419 (89i:53029)
- [17] N. Koiso, *On rotationally symmetric Hamilton's equation for Kähler-Einstein metrics*, Recent topics in differential and analytic geometry, Adv. Stud. Pure Math., vol. 18-I, Academic Press, 1990, pp. 327–337.
- [18] D. Petrecca, *On Sasaki-Ricci solitons and their deformations*, arXiv:1307.3410 [math.DG], 2013.
- [19] G. Tian and X. Zhu, *Uniqueness of Kähler-Ricci solitons*, Acta Math. **184** (2000), 271–305.
- [20] Y. Tsukamoto, *On Kählerian manifolds with positive holomorphic sectional curvature*, Proc. Japan Acad. **33** (1957), no. 6, 333–335.
- [21] X. Wang and X. Zhu, *Kähler-Ricci solitons on toric manifolds with positive first Chern class*, Adv. Math. **188** (2004), 87–103.

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